THE BESSELIAN FUNCTION,

BY PROF. ASAPH HALL.

The investigation in the theory of elliptic motion of the relations between the different anomalies, the expression of the radius-vector by means of series, and the expansion into series of various functions of the radius-vector and the anomalies, have led to interesting and valuable results. The theorem of Lagrange has here a direct application, but the method of determining the coefficients by means of definite integrals is more elegant. This method appears to have been first indicated by Poisson in the first edition of his Mecanique, but it was first completely worked out by Bessel in a memoir published among the memoirs of the Berlin Academy for 1816–17.

Let ε and g be the excentric and the mean anomalies, and e the excentricity of the ellipse, then

$$g = \varepsilon - e \sin \varepsilon.$$

Assume $\varepsilon - g = A_1 \sin g + A_2 \sin 2\, g + \dots + A_i \sin i\, g + \dots$ (1)

If i and i' are two whole numbers, we shall have when they are unequal

$$\int_o^\pi \cos i\, g \cos i'\, g . \, d\, g = 0, \quad \int_o^\pi \sin i\, g \sin i'\, g \, . \, d\, g = 0,$$

and when $i = i'$,

$$\int_o^\pi \cos^2 i\, g \, . \, d\, g = \tfrac{1}{2}\, \pi, \quad \int_o^\pi \sin^2 i \, . \, d\, g = \tfrac{1}{2}\, \pi.$$

Multiplying equation (1) by $\sin i\, g \, . \, d\, g$ and integrating, we shall have

$$A_i = \frac{2}{\pi} \int_o^\pi (\varepsilon - g) \sin i\, g \, . \, d\, g.$$

It is more convenient to use the variable ε, and integrating by parts, since $\varepsilon - g$ is zero at the limits o and π, we have

$$A_i = \frac{2}{i\pi} \int_o^\pi \cos i\, g \left(\frac{d\, \varepsilon}{dg} - 1 \right) . \, d\, g,$$

$$\text{or since} \int_o^\pi \cos i\, g \, . \, d\, g = 0,$$

$$A_i = \frac{2}{i\pi} \int_o^\pi \cos i\, g \, . \, d\, \varepsilon = \frac{2}{i\pi} \int_o^\pi \cos i\, (\varepsilon - e \sin \varepsilon) \, . \, d\, \varepsilon \, \dots \dots (2)$$

For the radius-vector we have a series proceeding according to the cosines of multiples of the mean anomaly, and the coefficients in this series can be derived from the value of A_i by a simple differentiation

with respect to the excentricity. In the series for the equation of center the coefficients can also be derived from the value of A_i, but by a more complicated relation.

In order to show another occurrence of this definite integral let it be required to change a series proceeding by sines and cosines of multiples of ε into one proceeding according to sines and cosines of multiples of g, a problem that occurs in the theory of perturbations. Here he have to solve the equations

$$\sin m\,\varepsilon = \Sigma\, A_i^m \sin i\,g; \qquad \cos m\,\varepsilon = \Sigma\, B_i^m \cos i\,g:$$

As before we have

$$A_i^m = \frac{2}{\pi}\int_o^\pi \sin m\,\varepsilon \sin i\,g\,.\,d\,g; \quad B_i^m = \frac{2}{\pi}\int_o^\pi \cos m\,\varepsilon \cos i\,g\,.\,d\,g.$$

Integrating by parts and omitting terms that are zero at the limits we have

$$A_i^m = \frac{2\,m}{i\,\pi}\int_o^\pi \cos i\,g \cos m\,\varepsilon\,.\,d\,\varepsilon; \; B_i^m = \frac{2\,m}{i\,\pi}\int_o^\pi \sin i\,g \sin m\,\varepsilon\,.\,d\,\varepsilon.$$

Putting for g its value, $\varepsilon - e\sin\varepsilon$, and then changing the products of the cosines and sines to sums and differences we find

$$A_i^m = \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i+m)\,\varepsilon - i\,e\sin\varepsilon\right].\,d\,\varepsilon + \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i-m)\,\varepsilon - i\,e\sin\varepsilon\right].\,d\,\varepsilon,$$

$$B_i^m = \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i-m)\,\varepsilon - i\,e\sin\varepsilon\right].\,d\,\varepsilon - \frac{m}{i\,\pi}\int_o^\pi \cos\left[(i+m)\,\varepsilon - i\,e\sin\varepsilon\right].\,d\,\varepsilon.$$

These integrals are of the same form as that in equation (2). Hence if we put

$$J_k^i = \frac{1}{\pi}\int_o^\pi \cos\,(i\,\varepsilon - k\sin\varepsilon)\,.\,d\,\varepsilon \quad \ldots\ldots\ldots\ldots\ldots (3)$$

we have

$$A_i^m = \frac{m}{i}\,(J_{ie}^{i+m} + J_{ie}^{i-m}); \quad B_i^m = \frac{m}{i}\,(J_{ie}^{i-m} - J_{ie}^{i+m}).$$

If therefore we have a table of the J function we can compute the coefficients easily. This function occurs also in the solution of the partial differential equations which are found in the theories of wave motion and of heat; and as it was first investigated and tabulated by Bessel it is called by the German mathematicians the Besselian function. More complete tables have been computed by Hansen, who has investigated this

function in his peculiar way, and who has brought out many curious properties.

We can find an equation between three successive values of the function as follows: Let

$$u = \sin (i\,\varepsilon - k \sin \varepsilon),$$

then

$$d\,u = i \cos (i\,\varepsilon - k \sin \varepsilon)\, d\,\varepsilon - \frac{k}{2} \cos \left[(i+1)\,\varepsilon - k \sin \varepsilon\right] d\,\varepsilon$$

$$- \frac{k}{2} \cos \left[(i-1)\,\varepsilon - k \sin \varepsilon\right] . \, d\,\varepsilon.$$

Since u is zero at the limits o and π we have by integrating this value of $d\,u$,

$$k\, J_k^{i-1} - 2\, i\, J_k^i + k\, J_k^{i+1} = 0. \quad \ldots\ldots\ldots\ldots\ldots (4)$$

This equation gives the value of the function from the values of the two lower orders, but it is not well adapted to numerical calculation, since it gives the value of a small quantity from the difference of two greater ones. It can however be easily transformed into a continued fraction well suited to such calculation.

From equation (3) we have

$$J_k^i = \frac{1}{\pi}\int_o^\pi \cos i\,\varepsilon \cos (k \sin \varepsilon) \,.\, d\,\varepsilon + \frac{1}{\pi}\int_o^\pi \sin i\,\varepsilon \sin (k \sin \varepsilon) \,.\, d\,\varepsilon.$$

And hence

$$J_k^o = \frac{1}{\pi}\int_o^\pi \cos (k \sin \varepsilon) \,.\, d\,\varepsilon = \frac{1}{\pi}\int_o^\pi d\,\varepsilon$$

$$\times \left\{ 1 - \frac{k^2 \sin^2 \varepsilon}{1\,.\,2} + \frac{k^4 \sin^4 \varepsilon}{1\,.\,2\,.\,3\,.\,4} - \frac{k^6 \sin^6 \varepsilon}{1.2.3.4.5.6} + \ldots\ldots \right\},$$

$$J_k^1 = \frac{1}{\pi}\int_o^\pi \sin \varepsilon \sin (k \sin \varepsilon) \,.\, d\,\varepsilon = \frac{1}{\pi}\int_o^\pi d\,\varepsilon$$

$$\times \left\{ k \sin^2 \varepsilon - \frac{k^3 \sin^4 \varepsilon}{1.2.3} + \frac{k^5 \sin^6 \varepsilon}{1.2.3.4.5} - \ldots\ldots \right\}.$$

Integrating the terms of the series between the limits o and π we have

$$J_k^o = 1 - \frac{k^2}{2^2} + \frac{k^4}{(2.4)^2} - \frac{k^6}{(2.4.6)^2} + \frac{k^8}{(2.4.6.8)^2} - \ldots\ldots$$

$$J_k^1 = \frac{2k}{2^2} - \frac{4k^3}{(2.4)^2} + \frac{6k^5}{(2.4.6)^2} - \frac{8k^7}{(2.4.6.8)^2} + \ldots\ldots$$

These values and equation (4) enable us to compute all values of the function. An elegant derivation of the general series for J_k^i will be found in Schlömilch's Compendium of the Higher Analysis, Vol. 2, p. 157.

This derivation depends on the following formula given by Jacobi, (Crelle, Vol. 15):

$$\int_o^\pi \psi(\cos t) \cos n\, t \,.\, d\, t = \frac{1}{1.3.5\ldots(2\,n-1)} \,.\, \int_o^\pi \psi^{(n)} (\cos t) \sin^{2n} t \,.\, d\, t,$$

where (n) denotes repeated differentiations.

DISCUSSION OF AN EXPONENTIAL CURVE.

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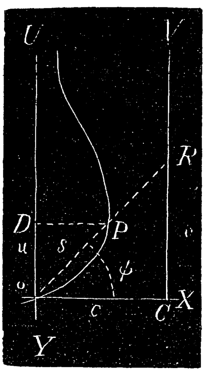
Construction by Points.—Let $O\,C = c$ be the base of any logarithmic system. On $O\,U$ and $C\,V$, perpendicular to $O\,C$, lay off $O\,D = u$, and $C\,R = v$, such that $u = \log v$ in the system whose base is c. The intersection of $O\,R$ and $D\,P$, (parallel to $O\,C$) fixes P, a point of the curve.



From the construction we can immediately derive the polar equation of the curve:

$$O\,P = s = \frac{u}{\sin\psi} = \frac{\log v}{\sin\psi} = \frac{\log . [c \tan . \psi]}{\sin\psi}$$

$$= \frac{\log . c + \log\tan\psi}{\sin\psi} \quad \ldots\ldots\ldots\ldots(1)$$

Substituting from $y = s \sin\psi$, and $\tan\psi = \frac{y}{x}$, we have for rectangular co-ordinates,

$$y = \log . c + \log\left(\frac{y}{x}\right) = \log . \left\{\frac{c\,y}{x}\right\} \text{ or } c^y = \frac{c\,y}{x},$$

whence we have the final equation of the curve,

$$x = \frac{y}{c^{y-1}}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and also successively,

$$\frac{dx}{dy} = c^{1-y}(1 - y\, l\, c),$$

$$\text{and } \frac{d^2 x}{d\, y^2} = c^{1-y}\, l\, c\,(y\, l\, c - 2); \text{ here } l\, c = \text{Nap. log} . c.$$

By reference to the equation (2) we see that if $c >$ unity the curve has the form indicated by our first diagram, which will give a maximum for x; also if $c =$ unity, (2) becomes $x = y$, the equation of a straight line bisecting the first angle, while if $c <$ unity, the curve passes to the other side of the line just mentioned, and is tangent to Y at $y = -\infty$, thus, and gives a minimum for x.